Instabilities in Pulse Propagation described by Generalized Nonlinear Schrödinger Equations

1st Shalva Amiranashvili
Weierstrass Institute for Applied Analysis and Stochastics
Berlin, Germany
shalva.amiranashvili@wias-berlin.de

2nd Uwe Bandelow
Weierstrass Institute for Applied Analysis and Stochastics
Berlin, Germany
uwe.bandelow@wias-berlin.de

3rd Raimondas Čiegis
Vilnius Gediminas Technical University
Vilnius, Lithuania
raimondas.ciegis@vgtu.lt

***Abstract*—This work investigates instabilities that can occur during pulse propagation along optical fibers. They can have physical origin, as the modulation instability, or being just numerical artefacts. We analyze in particular the numerical stability of operator splitting methods applied to Generalized Nonlinear Schrödinger Equations.**

***Index Terms*—nonlinear fibers; nonlinear Schrödinger equation; modulation instability; splitting methods**

I. Introduction

The propagation of optical pulses along nonlinear and dispersive optical fibers is often described by the generalized nonlinear Schrödinger equation (GNLSE) [1], which we present in the following form

$$i\partial_z\psi + \hat{D}\psi + \gamma|\psi|^2\psi = 0. \tag{1}$$

The GNLSE is suited to describe the evolution of stable pulses such as solitons, but also the modulation instability (MI) which is a fundamental physical physical phenomenon in the field of nonlinear waves [2]. MI occurs when small spontaneous modulations of initially uniform carrier wave begin to grow, leading to the emergence of various localized structures, such as robust solitary pulses [3] or spontaneous rogue waves [4].

Dispersive effects on wave propagation are governed in the GNLSE by the linear dispersion operator $\hat{D} = \sum_{j=2}^{J} \frac{\beta_j}{j!}(i\partial_\tau)^j$ and are separated from nonlinear effects, which are governed by the last term in Eq. (1). According to this separation, split-step methods provide an efficient approach for the numerical solution of the GNLSE, because they allow to separating complex problems into simpler, more manageable parts. In fact, in the context of optics, split-step methods are routinely used to study the propagation of pulses in optical networks [5], [6]. However, such methods are explicit and can lead to numerical instabilities, which we investigate. Results previously obtained for multiplicative splitting methods are extended to additive splittings [7]. An estimate of the largest possible integration step is derived and tested. The results are important when many solutions of GNLSE are needed, e.g., in optimization problems or statistical calculations.

We split the GNLSE into linear (A) and nonlinear (B) parts:

$$\partial_z\psi = (B + A)\psi \quad \Rightarrow \quad \psi(z + h) = e^{h(B+A)}\psi(z). \tag{2}$$

The above formal solution can be replaced by its splitting

$$\Psi(z+h) = e^{hB}e^{hA}\Psi(z); \quad e^{h(B+A)} = e^{hB}e^{hA} + O(h^2). \tag{3}$$

$\Psi(z)$ approximates $\psi(z)$ when the integration step $h \to 0$. The exponential $e^{h(B+A)}$ in Eq. (2) is understood as the propagator.

A. Multiplicative Methods

A multiplicative splitting $\mathcal{M}$ of order p with s stages, which involves $2s$ $a_{1\le n\le s}$ and $b_{1\le n\le s}$, is defined by

$$e^{h(B+A)} = \mathcal{M}(h) + O(h^{p+1}) \tag{4}$$

with $\mathcal{M}(h) = e^{b_s hB}e^{a_s hA} \cdots e^{b_1 hB}e^{a_1 hA}$. The splitting coefficients are selected so that the formal Taylor expansions of $\mathcal{M}(h)$ and $e^{h(B+A)}$ coincide as prescribed. The simplest are the Lie–Trotter splitting with $s = p = 1$ and the classical Strang splitting with $s = p = 2$ [8]. Another famous example with $s = p = 4$ is the Suzuki–Yoshida splitting [9], [10]. Equation (4) can be transformed to the equivalent form [10]

$$e^{h(B+A)+\Delta(h)} = e^{b_s hB}e^{a_s hA} \cdots e^{b_1 hB}e^{a_1 hA} \tag{5}$$

The leading term in the local error $\Delta(h) = O(h^{p+1})$ consists of commutators of length $p+1$. All commutators with shorter lengths must cancel each other, which gives a system of algebraic equations for the splitting coefficients. When a basis set in the space of commutators is chosen, the local error can be characterized by the ℓ^2 norm of the leading term in $\Delta(h)$,

$$\|\Delta(h)\| = \varkappa\frac{h^{p+1}}{(p+1)!} + O(h^{p+2}), \quad \varkappa = \text{const}. \tag{6}$$

The numerical value of $\varkappa$ is used to compare splittings of the same order to each other, and the final choice of $a_{1\le n\le s}$ and $b_{1\le n\le s}$ is made in favor of the minimal $\varkappa$.

B. Additive Methods

Additive splitting offers computational advantages, particularly for parallel architectures. If we use multiple splittings of the form (4) and then combine their predictions with appropriate weights, we obtain an additive splitting scheme.

Each multiplicative component of an additive splitting will be referred to as a thread, implying that different threads can be calculated independently on a multi-core machine before taking their weighted sum to accomplish an integration step. The simplest additive splitting is the second Strang splitting [8]

$$e^{h(B+A)} = \frac{1}{2}\left(e^{hB}e^{hA} + e^{hA}e^{hB}\right) + O(h^3). \tag{7}$$

Another example is the splitting with four threads and $p = 3$ derived by Burstein and Mirin [11]. As a third example, we follow the ARBBC splitting [12] with four threads and $p = 4$, which has a 10 times smaller local error parameter $\varkappa = 0.36$, compared to the Suzuki–Yoshida splitting. Both the accuracy and efficiency of the ARBBC splitting have been studied previously [12]. Moreover, Ref. [12] contains a detailed comparison between the aforementioned additive schemes and commonly used multiplicative schemes. However, to the best of our knowledge, little is known about the stability conditions for the additive splittings.

C. Stability Analysis

Both multiplicative and additive splitting methods are explicit schemes, and while they are easy to implement and very fast [13], they can suffer from numerical instabilities. An extension of earlier results on stability to the fourth-order Suzuki–Yoshida splitting was reported in [14], and to an arbitrary multiplicative splitting in [15], again in the GNLSE framework. The extension for the mentioned additive splitting methods are reported in [7]. To assess stability we have derived a *root condition* in [7], requiring eigenvalues of the numerical propagation matrix $\mathcal{M}$ to lie within the unit circle. This condition is sketched in Fig. 1, and it provides the stability domains. For GNLSE this gives the stability criterion for the stepsize h [16]:

$$h \le \frac{\pi}{\max_\Omega |M(\Omega)|}.$$

Here, $M(\Omega)$ is the even part of the dispersion function $D(\Omega)$.

As an example, have tested the correctness of splitting methods with respect to MI. In result, the second Strang splitting [8] fails regardless of step size h. The Burstein and Mirin splitting [11] requires, roughly speaking, a two times smaller integration step than the multiplicative splittings, but can capture the MI. The additive splitting proposed in Ref. [12] can be used with the same integration step as the multiplicative splittings. At least, the stability criterion should be an integral part of any implementation of splitting solvers for GNLSE. This is especially important when the dispersion function and, therefore, the differential operator in GNLSE are approximated by higher-order polynomials. The ARBBC splitting performs comparably to multiplicative schemes even at large h, but is both more accurate and less restrictive.

II. Conclusions

We have studied restrictions on the numerical integration step h that provide a numerically stable split-step solution of the GNLSE (1). The root condition provides a practical stability check and helps to avoid numerical artifacts. The technique has been applied to the mentioned additive splittings, which are of interest because different threads of such splitting schemes can be computed independently on a multicore machine, and ARBBC is the most reliable scheme.

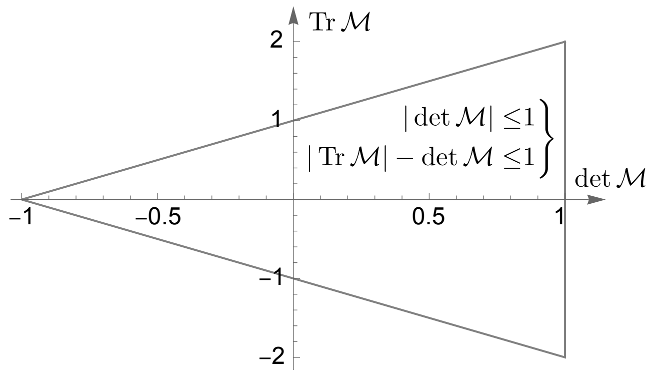

Fig. 1. Stability domain for the propagation matrix $\mathcal{M}$. If the point $(\det\mathcal{M},\ \mathrm{Tr}\,\mathcal{M})$ is inside the triangle, the eigenvalues of $\mathcal{M}$ are inside the unit circle. This is the root condition. From [7].

References

[1] G. P. Agrawal, *Nonlinear Fiber Optics*, Academic, New York, 4 ed., 2007.

[2] V. E. Zakharov and L. A. Ostrovsky, "Modulation instability: the beginning," *Physica D: Nonlinear Phenomena* **238**, pp. 540–548, March 2009.

[3] A. Hasegawa and M. Matsumoto, *Optical Solitons in Fibers*, Springer, 2003.

[4] M. Erkintalo, K. Hammani, B. Kibler, C. Finot, N. Akhmediev, J. M. Dudley, and G. Genty, "Higher-order modulation instability in nonlinear fiber optics," *Phys. Rev. Lett.* **107**, p. 253901, Dec 2011.

[5] S. Musetti, P. Serena, and A. Bononi, "On the accuracy of split-step Fourier simulations for wideband nonlinear optical communications," *Journal of Lightwave Technology* **36**, pp. 5669–5677, December 2018.

[6] J. de Negreiros Júnior, D. do Nascimento e Sá Cavalcante, J. de Moraes, L. R. Marcelino, F. de Carvalho Belchior Magalhães, R. R. Barboza, M. de Carvalho, and G. de Freitas Guimarães, "Ultrashort pulses propagation through different approaches of the split-step Fourier method," *Journal of Mechatronics Engineering,* **1**, pp. 2–11, December 2018.

[7] S. Amiranashvili, U. Bandelow, and R. Ciegis, "Stability of the Additive Splitting methods for the Generalized Nonlinear Schrödinger Equation," *Mathematics* **13**(8), p. 1301, 2025.

[8] G. Strang, "Accurate partial difference methods I: Linear Cauchy problems," *Arch. Rat. Mech. Anal.* **12**, pp. 392–402, January 1963.

[9] M. Suzuki, "Fractal decomposition of exponential operators with applications to many-body theories and Monte Carlo simulations," *Phys. Lett. A* **146**, pp. 319–323, June 1990.

[10] H. Yoshida, "Construction of higher order symplectic integrators," *Phys. Lett. A* **150**, pp. 262–268, November 1990.

[11] S. Z. Burstein and A. A. Mirin, "Third order difference methods for hyperbolic equations," *J. Comput. Phys.* **5**, pp. 547–571, June 1970.

[12] S. Amiranashvili, M. Radziunas, U. Bandelow, K. Busch, and R. Čiegis, "Additive splitting methods for parallel solutions of evolution problems," *Journal of Computational Physics* **436**, pp. 1–14, July 2021.

[13] M. Lovisetto, D. Clamond, and B. Marcos, "Integrating factor techniques applied to the Schrödinger-like equations. Comparison with Split-Step methods," *Applied Numerical Mathematics* **197**, pp. 258–271, March 2024.

[14] J. Yang, *Nonlinear Waves in Integrable and Nonintegrable Systems*, SIAM, 2010.

[15] S. Amiranashvili and R. Čiegis, "Stability of the higher-order splitting methods for the nonlinear Schr'odinger equation with an arbitrary dispersion operator," *Mathematical Modelling and Analysis* **29**(3), pp. 560–574, 2024.

[16] F. Severing, U. Bandelow, and S. Amiranashvili, "Spurious four-wave mixing processes in generalized nonlinear schrödinger equations," *J. Lightwave Technol.* **41**, pp. 5359–5365, August 2023.