Convergence of Factorization Rules for Modal Methods in Fourier-Bessel Basis. Is the Inverse Rule Really the Best?

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Abstract—**The choice of factorization rule can strongly affect the convergence of solutions to Maxwell equations based on the orthogonal expansion of electromagnetic fields. While this issue has already been investigated thoughtfully for the Fourier basis (plane-wave expansion), for other bases it has not yet received much attention. Although there are works showing that, in the case of the Fourier-Bessel basis (cylindrical wave expansion), use of an inverse factorization rule can provide faster convergence than Laurent's rule, these works neglect the fact that other rules are also possible. In this work, I demonstrate some different factorization rules for solving Maxwell equations in cylindrical coordinates using Fourier-Bessel expansion and I compare their convergence for a step-index fiber (which has a known exact solution and thus enables the absolute numerical error to be determined), as well as for several VCSEL structures. This allows to identify the factorization rule that gives the fastest convergence of the modal method using the Fourier-Bessel basis.**

Numerical methods based on modal expansion are popular for the analysis of optical properties of photonic devices. Their common factor is that the computational domain is divided into a stack of connected waveguide layers (usually uniform in the propagation direction) and, in each of these sections, the electromagnetic field is represented as a linear combination of its eigenmodes. These eigenmodes can be determined either rigorously [1], [2], [3] or approximately. In the first case—known as the classical modal method—the eigenfunctions are found by calculating zeros of a transcendental equation. However, such equation can be usually constructed for simple geometries with no continuous refractive index variation (e.g. due to the temperature distribution) and its solution requires searching for the zeros of the transcendental equation in the complex plane, if the structure has loss or gain. Hence, another approach of representing such eigenmodes is either with finite differences [4] or an expansion in a set of orthogonal functions defined globally in all investigated layers. The choice of such orthogonal basis leads to different variants of the modal method. For example, such basis can form a Fourier series, as in very popular rigorous coupled-wave analysis (RCWA) [5], [6], [7], or a Bessel-Fourier series [8], [9], [10], [11], [12]. The former is a natural choice for periodic structures in the Cartesian coordinate system, whereas the latter is the best option in cylindrical coordinates. Other possibilities are Hermite-Gauss functions [13], [14] or wavelets [15], [16], [17], [18].

Globally defined basis has a significant drawback when analyzing structures with sharp material boundaries that cause the discontinuities in the electric field. Approximation of such field with continuous basis functions results in unwanted Gibbs phenomenon. This can be remedied by a proper choice of factorization rules, what has been extensively studied and is well understood in case of Fourier basis in Cartesian coordinates [19], [20], [21], [22]. Several authors have presented and successfully implemented similar work for the Fourier-Bessel basis[23], [11]. However, the Fourier-Bessel basis is not algebraically as straightforward as the Fourier one. For this reason, there is more than one way to implement the inverse rule.

For any rule, the radial and angular components of the electric field are expanded in the Fourier-Bessel basis as follows

$$\begin{aligned} E_r(r,z) &= \big|k_\ominus\big\rangle\, E_s^k \,+\, \big|k_\ominus\big\rangle\, E_p^k, \\ E_\varphi(r,z) &= \big|k_\ominus\big\rangle\, E_s^k - \big|k_\oplus\big\rangle\, E_p^k \\ E_z(r,z) &= \big|k\big\rangle E_z^k \end{aligned}$$

and the magnetic field is expanded similarly. In the above equations m is the angular mode number, k radial wavevector and $\big|k_\ominus\big\rangle$, $\big|k\big\rangle$, and $\big|k_\ominus\big\rangle$ denote expansion in the J_{m-1}, J_m, and J_{m+1} basis, respectively, where J_μ is the Bessel function of the first kind. The vertical component can be expanded with direct or inverse rule as follows

$$E_z^k = -\frac{i}{k_0}\left\langle k\middle|\varepsilon_{zz}^{-1}\middle|k'\right\rangle [k'] \left(H_p^{k'} + H_s^{k'}\right) \qquad \text{— direct or}$$

$$E_z^k = -\frac{i}{k_0}\left\langle k\middle|\varepsilon_{zz}\middle|k'\right\rangle^{-1} [k'] \left(H_p^{k'} + H_s^{k'}\right) \qquad \text{— inverse rule.}$$

The inverse expansion of E_z yields the most precise results. The different situation is with E_r and E_φ. They may be exp

I compare the modal method convergence in three cases: direct factorization rule for all electric field components (E_r, E_φ, and E_z), semi-inverse rule, where only E_z is factorized according to the inverse rule, and full inverse rule used for E_z and E_r. In theory, the last case should provide the best convergence [11]. However, in cylindrical coordinates, one does not represent radial and angular components of the electric field (E_r and E_φ, respectively) directly, but as $E_r = E_s + E_p$ and $E_\varphi = E_s - E_p$, where both E_s (expanded in J_{m-1} basis, where J is a Bessel function of the first kind) and E_p (expanded in J_{m+1}) are discontinuous at the boundaries. This fact strongly decreases the convergence of the full inverse rule.

This can be seen by analysis of the convergence rate of a step-index waveguide with three different factorization rules (Fig. 1). Because in such simple case the exact solution is known, it is possible to determine the absolute error in each case. The lowest error for both HE_{11} and EH_{11} modes is obtained with the semi-inverse rule, i.e. when the vertical component E_z is factorized with the inverse

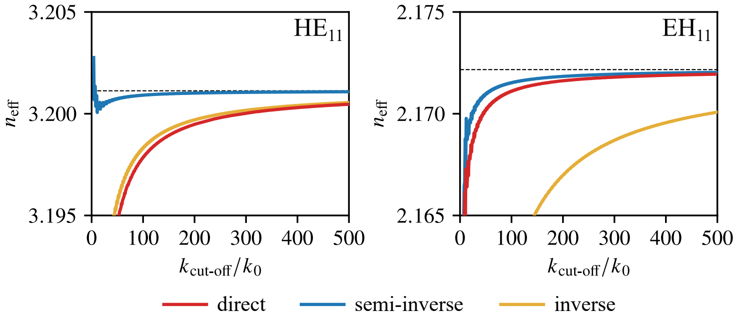

Figure 1. Comparison of convergence for three factorization rules of HE$_{11}$ and EH$_{11}$ modes in simple cylindrical step-index waveguide. Black dashed lines indicate the exact solution.

rule, whereas both E_r and E_φ are calculated using the direct rule. In both cases, the inverse rule yields the worst convergence, which may be surprising.

In order to better test the applicability of the investigated factorization rules, I now compute the resonant wavelength and the threshold gain of a simple step-gain-profile VCSEL [24]. Contrary to the simple step-index waveguide discussed in the previous section, there is no exact method for computing the values of these parameters. For this reason, I am only able to compare my results with either experimental data or with results obtained by different numerical methods. The considered factorization rules yield consistent results. Surprisingly, in all cases for the large aperture, the threshold gain is more consistent than the wavelength. This can be explained by the fact that there is very little scattering and the dominant losses are attributed to emission in the vertical direction, which is only slightly influenced by the differences in the lateral expansion of the modes. However, the resonant wavelength stabilizes fastest when direct and semi-inverse rules are used and the inverse rule yields a different wavelength.

I the talk, I consider also mode advanced designs, like an antiresonant VCSEL with an oxide island [25], [26]. The general conclusion is that I recommend to always apply the semi-inverse rule in calculations based on the modal method using Fourier-Bessel expansion in cylindrical coordinates.

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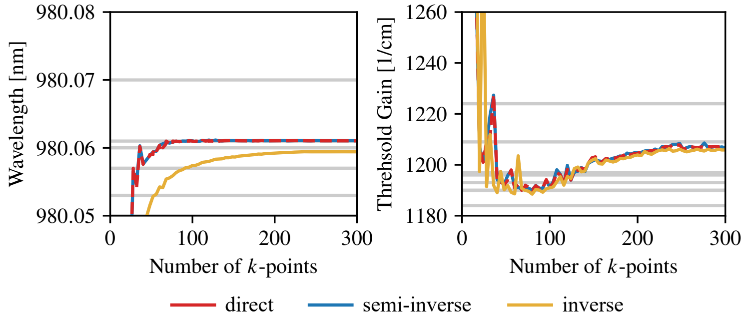

Figure 2. Convergence of the resonant wavelength and threshold gain of the HE$_{11}$ mode in a VCSEL with 8-μm aperture. Gray horizontal lines indicate the results obtained by other authors [24].

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