Estimation of Frequency Noise Characteristics and Data-Driven Modeling of Narrow-Linewidth Semiconductor Lasers

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***Abstract*—The design of narrow-linewidth lasers requires stochastic laser models providing a realistic description of the noise in the device. We present a statistical inference approach to extract the frequency noise characteristics and model parameters of narrow-linewidth lasers from delayed self-heterodyne beat note experiments. By exploiting prior knowledge about the statistical distribution of the measurement data, accurate estimates of the parameters of the free running laser can be achieved even in the presence of considerable detector noise. The approach is demonstrated for simulated time series data using a stochastic laser rate equation model including $1/f$–type noise.**

I. Motivation

Narrow-linewidth lasers are core elements of emerging quantum technologies such as optical atomic clocks, matter-wave interferometers and quantum computers based on ion traps or neutral atoms in optical lattices. These applications require highly coherent light with a narrow intrinsic (Lorentzian) linewidth [1], which is typically broadened by additional $1/f$–type noise (also flicker noise). Because of this non-Markovian noise, the laser linewidth depends on the measurement time such that a detailed characterization of the spectral quality of the laser requires the measurement of the frequency noise power spectral density (FN–PSD).

For the theoretical design of improved low-noise devices, stochastic laser models are required that provide an accurate description of the fluctuation characteristics matching experimental observations. A particular challenge is the quantitatively correct inclusion of non-Markovian noise that generates characteristic frequency drifts and inhomogeneous broadening of the lineshape. We strive for a data-driven modeling approach using time series data from delayed self-heterodyne (DSH) beat note measurements, see Fig. 1 (a), which is the standard experimental technique to determine the laser linewidth [2]. The method allows to extract the FN–PSD and numerous model parameters from time series data, but a direct analysis of the measured data is not trivial as both the footprint of the interferometer as well as the detector noise must be taken into account in order to reconstruct the noise characteristics of the free-running laser. In this paper, we describe a Bayesian inference approach involving a Markov-chain Monte Carlo (MCMC) method, to estimate model parameters from DSH experiments by taking all important aspects of the measurement process into account [3].

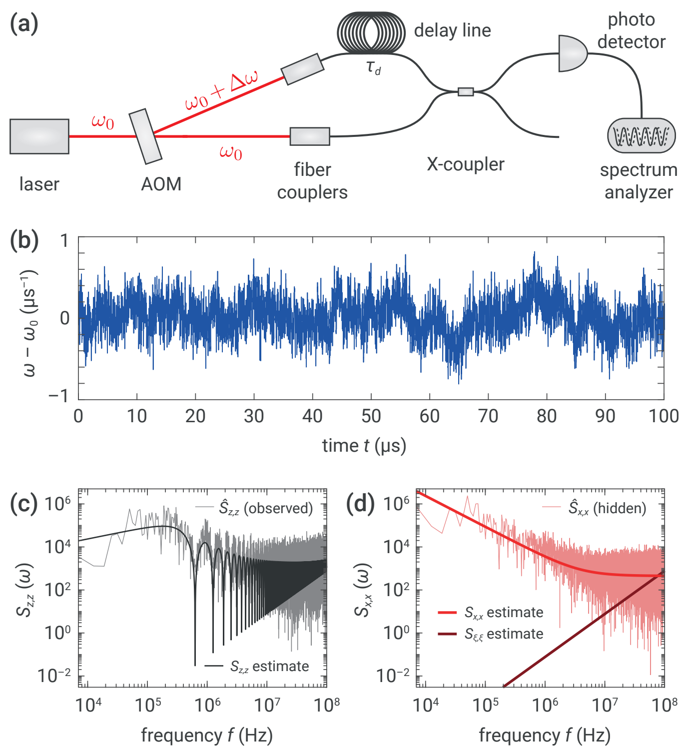

Fig. 1. **(a)** Experimental setup of the DSH method, reprinted with permission from [4]. **(b)** Time series of frequency fluctuations around the continuous wave frequency (moving average over 50 ns). Characteristic frequency drifts are induced by the colored noise in Eq. (1). **(c)** Inferred power spectral density $S_{z,z}(\omega)$ estimated using the MCMC method along with the periodogram $\hat{S}_{z,z}(\omega)$ of a single observed time series $z(t)$. **(d)** Estimated FN–PSD $S_{x,x}(\omega)$ of the laser and periodogram of the hidden time series $x(t)$.

II. Stochastic Laser Rate Equations

We aim for parameter identification and estimation of the FN–PSD of a single-mode semiconductor laser obeying the stochastic laser rate equations (Langevin equations) [4]

$$\begin{aligned} \dot{P} &= -\gamma\left(P-P_{\mathrm{th}}\right)+G\left(P,N\right)P+G_{\mathrm{sp}}\left(P,N\right)+F_{P}\left(t\right), \\ \dot{\phi} &= \frac{1}{2}\alpha_{H}G\left(P,N\right)+F_{\phi}\left(t\right), \\ \dot{N} &= \frac{\eta I}{q}-R\left(N\right)-G\left(P,N\right)P-G_{\mathrm{sp}}\left(P,N\right)+F_{N}\left(t\right), \end{aligned} \tag{1}$$

where P is the number of photons, ϕ is the optical phase and N is the number of carriers in the active region. Moreover, γ is the optical loss rate, P_{th} is the thermal photon number, G describes the net-gain function, G_{sp} is the rate of spontaneous emission into the lasing mode, α_H is the linewidth enhancement factor, η is the injection efficiency, I is the pump current and R is the rate of non-radiative recombination and spontaneous emission into waste modes. The Langevin noise sources $F_{P,N,\phi}(t)$ include next to the (commonly considered) white noise part [1] also colored noise contributions with a $1/f^{\nu}$–type frequency dependency that give rise to characteristic frequency drifts, see Fig. 1 (b). Further details on the model equations including correlation functions and parameter values are given in Ref. [4].

III. Bayesian Parameter Estimation

The stochastic rate equation model (1) generates a stationary Gaussian process with time-correlated noise, for which numerous parameter estimation approaches exist. Most prominent is the maximum likelihood estimation method applied on time domain data, which is however computationally intractable in the present case because of the typically long interferometer delay τ_d and the long-time correlations due to the $1/f$ noise. This requires both the consideration of long time series and a high-dimensional Markovian embedding, which makes the construction and maximization of the likelihood function very expensive. These limitations can be overcome by formulating the estimation procedure in the frequency domain, but this requires knowledge of the expected statistical distribution of the periodogram data, which is no longer Gaussian distributed.

In the case of FN–PSD reconstruction [3, 4], the measured signal of the DSH experiment can be described by

$$z(t) = (h * x)(t) + \xi(t), \tag{2}$$

where $z(t)$ is the observed time series, $h(t) = \delta(t) - \delta(t - \tau_d)$ is the transfer function of the interferometer (convolution kernel), $x(t)$ is the hidden time series of the frequency fluctuations of the laser (*i.e.*, $x(t) \mathrel{\hat{=}} \omega(t) - \omega_0 = \dot{\phi}(t)$, see Fig. 1 (b)) and $\xi(t)$ is additive detector noise. Our goal is to characterize the statistical properties of $x(t)$. Fourier transform of Eq. (2) yields a relation for the PSDs

$$S_{z,z}(\omega) = |H(\omega)|^2 S_{x,x}(\omega) + S_{\xi,\xi}(\omega), \tag{3}$$

where $S_{z,z}(\omega)$ is the PSD of the observed data and the transfer function is $H(\omega) = 1 - \exp(i\omega\tau_d)$. We assume the following models for the hidden signal and the detector noise PSD

$$S_{x,x}(\omega) = \frac{C}{|\omega|^{\nu}} + S_{\infty}, \qquad S_{\xi,\xi}(\omega) = \sigma^2\omega^2, \tag{4}$$

where S_{∞} is the intrinsic linewidth and the parameters C and ν quantify the flicker noise. We seek to estimate parameters from a single time series, which requires prior knowledge about the statistical distribution of the corresponding periodogram data. Transformation of random variables shows that the periodogram of a Gaussian time series is exponentially distributed

$$\hat{S}_{z,z}(\omega) \sim \text{Exp}(\lambda(\omega)), \quad S_{z,z}(\omega) = \langle \hat{S}_{z,z}(\omega) \rangle = \lambda^{-1}(\omega),$$

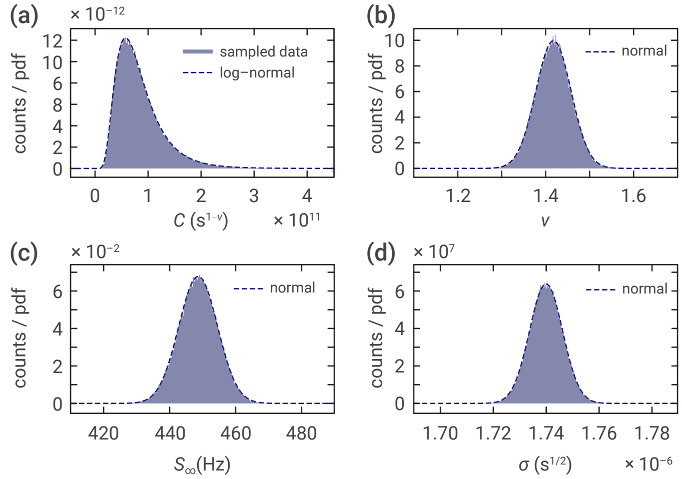

Fig. 2. Histograms of parameter estimates (marginal probability distributions) characterizing the FN–PSD in Eq. (4) obtained using the MCMC method. The sampled distributions are shown along with normally (Gaussian) and log-normally distributed probability density distributions (pdfs).

where the parameter $\lambda(\omega)$ is identified with the inverse expectation value $\lambda(\omega) = S_{z,z}^{-1}(\omega)$ using the model PSDs (4) at each frequency ω. We employ a Markov-chain Monte Carlo (MCMC) method (Metropolis–Hastings algorithm) [5] to infer on the most probable set of parameters underlying a given time series by maximizing the corresponding likelihood function. Notably, the method provides a quantification of the estimation uncertainty right away, see Fig. 2 .

IV. Summary

Bayesian inference enables the estimation of unknown parameters from noisy measurement data. The method thus can be used to improve the evaluation of experiments and to identify hardly accessible parameters for dynamical models. We demonstrate the approach using time series data from DSH measurements on narrow-linewidth lasers and extract parameter estimates to improve the modeling of noise sources in lasers.

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References

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