Data-Driven Modeling of Non-Markovian Noise in Semiconductor Lasers

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***Abstract*—Non-Markovian noise degrades the coherence properties of semiconductor lasers and contributes significantly to broadening of the linewidth. Since modeling of such colored noise systems from first principles is not accessible, we aim for a data-driven modeling approach in which a system of stochastic rate equations shall be reconstructed from time series data.**

I. Introduction

Semiconductor laser diodes are core elements of many important technological applications including coherent communication, optical sensing, frequency metrology and spectroscopy. Central to these applications is the spectral coherence of the emitted radiation, which is quantified by its frequency fluctuation spectrum or alternatively the laser's spectral linewidth. In the ideal quantum-limited case, the fluctuations in the optical field are dominantly driven by spontaneous emission of photons into the laser mode, which is well described as white noise and gives rise to the well-known (modified) Schawlow–Townes formula for the linewidth. In real devices, however, additional noise sources come into play (*e.g.*, recombination noise, absorption fluctuations, thermal noise, mechanical vibrations etc.), which can lead to a considerable reduction of the spectral coherence. A prominent feature observed in experimental noise spectra is a power-law type behavior dominating the noise at low frequencies ($1/f$ noise, "flicker noise"), whose origin is not yet fully understood. Such colored noise contributions are characterized by stochastic processes with correlated increments and in general lead to a broadening of the laser's spectral linewidth. A proper understanding of the noise in semiconductor lasers is therefore crucial for engineering low-noise devices.

In this paper, we explore the perspectives of a data-driven modeling approach using time-series regression techniques to infer on stochastic rate equation models from measurement data. To this end, we will first develop the methodology using simulation data (based on a stochastic model described in the following).

II. Stochastic Laser Rate Equations

The standard theory of noise in semiconductor lasers is based on a quantum mechanical treatment of the light-matter interaction in an open quantum system described by Heisenberg–Langevin equations [1]. The presence of dissipative processes in the open system (*e.g.*, carrier injection, cavity losses, recombination, polarization decay etc.) necessitates the introduction of suitable noise operators in order to preserve the canonical commutator relations (fluctuation–dissipation theorem). In the case of centered Gaussian processes, the noise is entirely characterized by its covariance matrix, which is closely related with the diffusion matrix. In the case of Markovian noise (instantaneous processes without memory), the latter can be obtained from first principles via the time-dependent Einstein relation [2]. Passing to the semi-classical limit, one arrives at a set of stochastic rate equations (Langevin equations) [3], which provide a consistent description of the fluctuation characteristics of the laser.

Here we consider a single-mode laser diode described by a set of Itô-type stochastic differential equations for the number of intra-cavity photons P, the optical phase ϕ and the number of carriers N in the active region given as

$$\begin{aligned} \mathrm{d}P = {} & \left(\left(\Gamma v_g g - \frac{1}{\tau_p}\right) P + \Gamma v_g g_{\text{sp}}\right) \mathrm{d}t \\ & + \sqrt{\frac{P}{\tau_p}}\,\mathrm{d}W^P_{\text{loss}} + \sqrt{\Gamma v_g g_{\text{sp}} P}\,\mathrm{d}W^P_{\text{st–em}} \\ & + \sqrt{\Gamma v_g g_{\text{abs}} P}\,\mathrm{d}W^P_{\text{st–abs}} + \sqrt{\Gamma v_g g_{\text{sp}}}\,\mathrm{d}W^P_{\text{sp}}, \end{aligned} \tag{1}$$

$$\begin{aligned} \mathrm{d}\phi = {} & \frac{\alpha_H}{2}\left(\Gamma v_g g - \frac{1}{\tau_p}\right)\mathrm{d}t \\ & + \frac{1}{2P}\Bigg(\sqrt{\frac{P}{\tau_p}}\,\mathrm{d}W^\phi_{\text{loss}} + \sqrt{\Gamma v_g g_{\text{sp}} P}\,\mathrm{d}W^\phi_{\text{st–em}} \\ & \qquad + \sqrt{\Gamma v_g g_{\text{abs}} P}\,\mathrm{d}W^\phi_{\text{st–abs}} + \sqrt{\Gamma v_g g_{\text{sp}}}\,\mathrm{d}W^\phi_{\text{sp}}\Bigg), \end{aligned} \tag{2}$$

$$\begin{aligned} \mathrm{d}N = {} & \left(\frac{I}{q} - r\left(N\right) - \Gamma v_g g_{\text{sp}} - \Gamma v_g g P + F\right)\mathrm{d}t \\ & + \sqrt{\frac{I}{q}}\,\mathrm{d}W_I + \sqrt{r\left(N\right)}\,\mathrm{d}W_R - \sqrt{\Gamma v_g g_{\text{sp}}}\,\mathrm{d}W^P_{\text{sp}} \\ & - \sqrt{\Gamma v_g g_{\text{sp}} P}\,\mathrm{d}W^P_{\text{st–em}} - \sqrt{\Gamma v_g g_{\text{abs}} P}\,\mathrm{d}W^P_{\text{st–abs}}. \end{aligned} \tag{3}$$

Here, v_g is the group velocity, Γ is the optical confinement factor, τ_p is the photon lifetime, α_H is the linewidth enhancement factor and I is the pump current. The (net-)gain is modeled as

$$g = \frac{g_0 N_{\text{tr}}}{1 + P/P_{\text{sat}}} \log\left(N/N_{\text{tr}}\right),$$

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where $N_{\rm tr}$ is the transparency density and $P_{\rm sat}$ corresponds to the gain compression coefficient. Following [4], the spontaneous emission coefficient is well described by

$$g_{\rm sp} = \frac{1}{2}\frac{g_0 N_{\rm tr}}{1 + P/P_{\rm sat}} \log\left(1 + (N/N_{\rm tr})^2\right),$$

which provides the correct asymptotics and does not require any additional parameters. Note that this model for $g_{\rm sp}$ avoids the introduction of the population inversion factor and tacitly implies a state-dependent $\beta_{\rm sp}$-factor. The absorption coefficient follows as $g_{\rm abs} = g_{\rm sp} - g$. Non-radiative recombination and spontaneous emission into waste modes are described by $r(N)$, which is modeled in the usual way (ABC model).

The system (1)–(3) includes numerous stochastic processes (associated with the respective dissipation) each modeled by a standard Wiener processes W_k (Brownian motion). The stochastic increments $\mathrm{d}W_k$ describe Gaussian white noise with infinitesimal variance $\mathrm{d}t$. All white-noise sources are independent such that $\mathrm{d}W_k \mathrm{d}W_l = \delta_{k,l}\,\mathrm{d}t$ by Itô's lemma [5]. In addition to white noise, our model includes a single colored noise source $F(t)$ entering Eq. (3), which acts as the origin of $1/f$ noise due to index and absorption fluctuations, cf. [6].

III. Non-Markovian Noise

One of the major difficulties in understanding $1/f$ noise is the inability of simple physical models to produce a $1/f$ spectrum in a natural way. In spite of the omnipresence of this phenomenon and almost a century of research on $1/f$ noise, there is no commonly accepted theory explaining such noise.

Several modeling approaches exist, which give a $1/f^{\alpha}$-type spectrum over a sufficiently large frequency range. The standard approach [7] is based on a superposition of Ornstein–Uhlenbeck fluctuators

$$\mathrm{d}x_k = -\gamma_k x_k\,\mathrm{d}t + \sqrt{2\gamma_k}\,\mathrm{d}W_k,$$

where the distribution of the decay rates γ_k follows a power-law. The fluctuators are combined as $F(t) = \sigma(N)\sum_k x_k(t)$, where $\sigma(N)$ is adjusted such that the model obeys Hooge's empirical law on $1/f$ noise [7]. Alternatively, nonlinear stochastic differential equations [8] or fractional Brownian motion [9] also represent viable modeling approaches.

IV. Regression

We consider a laser operating in continuous wave mode, where the fluctuation dynamics is well described by a linearization of the system (1)–(3) at the steady state. Typical measurement data for the optical field dynamics of such a laser are obtained from self-heterodyne beat note measurements, which give access to time series of the power and the phase. We eliminate the equation for the carrier number (which is not directly observable), by formally solving the linearized version of Eq. (3) using its Green's function. Substituting the resulting expression into the remaining equations for the photon number and phase yields a system of memory-type equations that are non-local in time in both the deterministic and the stochastic part. Our goal is to infer on the parameters of the stochastic system by performing a regression analysis using an ARIMA (p,d,q) type model [10], see Fig. 1, where the depth of the memory kernels (and the number of parameters) depends on the size of p and q. Given sufficiently long time series, the model parameters can be estimated with sufficient precision and thus the stochastic system (and the underlying noise model) can be reconstructed to a large extent from data.

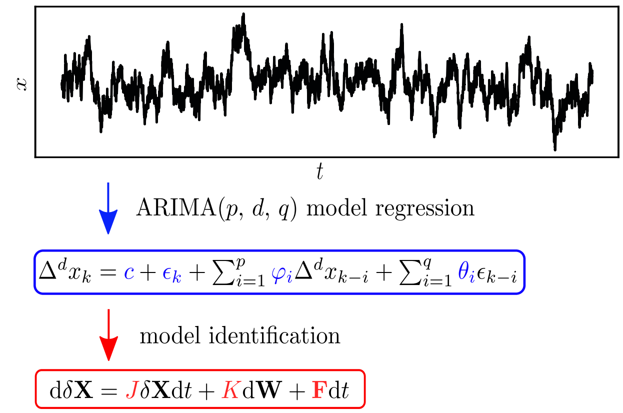

Fig. 1. Statistical inference of the model parameters from times series data (optical power and phase) using an ARIMA (p,d,q) model.

V. Outlook

The regression technique developed here for the case of single-mode stochastic laser rate equations shall be extended towards lasers with time-delayed feedback (stochastic Lang–Kobayashi equations) or even partial differential equation systems (stochastic traveling wave model) in the future. Furthermore, after having gained a proper understanding of the limitations from working with simulated data, the method shall be applied to experimental time series.

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